Error Propagation in Dense Wireless Networks with Cooperation *

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ABSTRACT

We study error propagation in wireless cooperative broadcast protocols. In our model, nodes in the network are randomly deployed in a fixed region. The message from a certain source node is relayed by multiple groups of cooperating relays that are located in predetermined consecutive swaths of the network area. We refer to these groups of relays as "levels". Our analysis is based on the derivation of recursive equations that express the error rate at each relay in a given level as a function of the error probabilities of the nodes in the previous levels. To provide analytical results we take the limit as the number of nodes goes to infinity while the relay power goes to zero, so that the relay power density per unit area is constant. In the limit, we show that the relay power density and area of the regions that correspond to the levels lead to fixed points in the error rate that prevent catastrophic error propagation, regardless of the distance the message is transmitted over.

Categories and Subject Descriptors

C.2.1 [**Network Architecture and Design**]: Wireless communication

General Terms

Algorithms, Performance, Reliability

Keywords

Asymptotic analysis, Cooperative communication, Error propagation, Wireless sensor networks

1. INTRODUCTION

Several papers [1–3], have recently highlighted the benefits of cooperative transmission for some network operations like flooding [4] which are important in sensor networks. In [5–7] it is shown that cooperation at the physical layer can provide significant power advantages. In particular, in [5, 6] it was noted that cooperative broadcast in dense networks requires a critical power density (total power transmitted per unit area) asymptotically. If the message is transmitted with a power greater than this critical power density then it is guaranteed to reach all network nodes. The other interesting aspect of the analysis in [5, 6] is that it shows how the message is passed from one level of cooperating nodes to the next, by covering a finite region of the network at each passage. The analysis, however, does not address the issue of error propagation because the packets are assumed to be long enough that the error can asymptotically be set to zero.

Not many authors have ventured into the analysis of the error propagation phenomenon in large cooperative networks. Several authors avoided the issue by invoking information theoretic limits [2, 4]. Others only considered small scale networks [3, 8]. In all cases the expressions are challenging to interpret. On the other hand, it is quite easy to obtain scaling laws for non-cooperative multi-hop systems (see Section 3.3) and to show that the probability of error degrades steadily with the number of hops and, thus, with the distance covered by the message. The obvious question to ask is if this is also the case in cooperative broadcasting and if cooperative broadcasting has worse error propagation. Intuition seems to be of little help because in cooperative broadcasting erroneous signals are mixed with the correct ones at the physical layer. To answer this question, we consider two cooperative broadcast protocols similar to the ones considered in [5, 6] (see Section 2) for the case of uncoded binary transmission. In both protocols contiguous groups of nodes relay a BPSK modulated message from a source at one edge of the network to all other nodes in steps. One scheme allocates orthogonal channels for all the nodes that cooperate while the second scheme forces all nodes in each group to share the same channel. Using techniques first introduced in [5] for cooperative networks, we analyze the error propagation by using the limit distribution of the received signal in the asymptotic regime i.e. the number of nodes that cooperate approaches infinity, while their individual relay power goes to zero so that the power emitted per unit area is constant.

Most papers that discuss cooperative schemes focus on performance gains in terms of diversity. Our analysis instead reveals that there are additional advantages that stem

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from employing cooperation at the physical layer rather than using the *collision channel* model of multi-hop networks or the orthogonal relay channels that most other cooperative schemes advocate. We show that if transmissions are added up at the physical layer, erroneous transmissions do not necessarily lead to catastrophic error propagation and that as the network size grows, the average error can be controlled precisely by controlling the size of the cooperative groups and the power density used in relaying the message. Furthermore, a bandwidth expansion is not necessary to attain this advantage. The required bandwidth can be kept finite by using asynchronous cooperation schemes such as [7] and [9].

The main results of this paper are as follows:

1. We derive equations that allow us to evaluate the average BER of uncoded transmission in a dense wireless network with two different cooperative protocols.

2. We prove that *asymptotically, both cooperative protocols have a bounded average BER at a sufficiently high SNR, regardless of the distance covered by the transmission.* Interestingly, the bounds are $\approx Q(\sqrt{SNR})$ for the orthogonal channel scheme and $\approx 1/SNR$ for the non-orthogonal one.

3. We compare these schemes with each other and with a non-cooperative multi-hop scheme that covers the same distance with the same total power resources. We argue that the cooperative schemes provide better error performance.

2. SYSTEM MODEL

Consider a source node that wants to broadcast its message to a certain predetermined region of the network. Assume that N nodes are deployed with a random uniform distribution in this region of unit area. For simplicity, we will assume that the region is a strip of width W and length L ($WL = 1$) and that the source node located at one edge of the strip transmits with power P_s. We also assume that the source and relay data are BPSK modulated. The network region is divided into consecutive areas L_k, $k = 1, \dots, K$. For the simple strip model considered:

$$L_k = \{(k-1)L/K \leq x < kL/K, |y| \leq W/2\}, k = 1, \dots, K$$

while $L_0 = (0, 0)$ is only one point. All nodes that fall within L_k form a *level* k set

$$\mathcal{S}_k = \{i : (x_i, y_i) \in L_k,\ i = 1, \dots, N\}, \quad k = 1, \dots, K.$$

The source is the only node in level zero i.e. $\mathcal{S}_0 = \{i = 0, (x_0, y_0) = (0, 0)\}$. For our analysis we consider two cases: a) the case where the cooperative nodes in each level use *orthogonal channels*, such as Orthogonal Space Time Codes (OSTC), or simply TDMA/FDMA or CDMA scheduling; b) the case where each level uses only one signal dimension and hence the transmissions are *non-orthogonal*.

Clearly, in the first case we need a bandwidth expansion proportional to the number of nodes in the network, in all cases except OSTC, because we need $|\mathcal{S}_k|$ channels per level and each level transmits over a different time slot. Instead, in the second case, we only require a bandwidth expansion on the order of the number of levels K, which is comparable to what a non cooperative multi-hop system requires. In our model we also assume that all nodes in each level $\mathcal{S}_k$ transmit synchronously at the symbol level. Since the events occur in a chain, synchrony at the symbol level can be enforced if the transmission is narrow-band and by having each level estimate the time of arrival of the previous level packet and retransmit within a deadline from its arrival time. Small scale asynchronism is incorporated in our model because we assume the presence of fading. In either case, we assume that the nodes can detect whether the previous level transmitted a message and what the link gain of each channel is from a preamble[1].

3. ORTHOGONAL TRANSMISSIONS

3.1 Random Finite Network

Assume that the orthogonal channels experience random flat fading and the link coefficient is $\beta_{ji} \triangleq \alpha_{ji} e^{j\theta_{ji}} \sqrt{P_r l(d_{ji})}$ where P_r is the power of the relay transmissions. The fading is characterized by deterministic large scale fading factors $l(d_{ji})$ which are functions of the internode distance (d_{ji}) only, envelopes α_{ji} that may or may not be independent and random phases θ_{ji} that are i.i.d. uniform in $[0, 2\pi]$. The latter assumption is justified by the fact that nodes have independent oscillators and separate RF front-ends. The network is pre-partitioned as described in the previous section and we assume that the source transmits a BPSK message with power P_s. Then, for each symbol b transmitted by the source, the discrete time baseband complex equivalent model for the corresponding received symbol at the j^{th} node in $\mathcal{S}_1$ is:

$$r_j^{(1)} = b\alpha_{j0} e^{j\theta_{j0}} \sqrt{P_s l(d_{j0})} + w_j, \qquad (1)$$

where α_{j0} is the small scale fading factor at node j, θ_{j0} is the carrier phase shift, w_j is a circularly symmetric complex Gaussian noise sample with zero mean and variance N_o, d_{j0} is the distance between the source and the j^{th} node, and $l(d_{j0})$ is the path-loss attenuation function which we assume to be deterministic (e.g. for the free-space model it is $1/d_{j0}^2$). Because we assume that the channel gain has been estimated perfectly by all nodes in the level, the nodes in $\mathcal{S}_1$ can use a coherent detector and the Bit Error Rate (BER) at a node j in $\mathcal{S}_1$ is given by

$$P_e^{(1)} = Q\left(\sqrt{\frac{2P_s}{N_o} |\alpha_{j0}|^2 l(d_{j0})}\right). \qquad (2)$$

where the Q-function is defined as $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{t^2}{2}} dt$. As specified by the cooperative transmission protocol in Section 2, the relay nodes in $\mathcal{S}_1$ will transmit their decoded bits over orthogonal channels. Subsequently, the relay nodes in all other levels will do the same. But not all relay nodes will decode the bit they received correctly, therefore errors will be introduced into the bit retransmission flow at the physical layer. Let the orthogonal channel between the relay nodes $i \in \mathcal{S}_k$ and $j \in \mathcal{S}_{(k+1)}$ have a channel gain $\beta_{ji} \triangleq \alpha_{ji} e^{j\theta_{ji}} \sqrt{P_r l(d_{ji})}$. The received signal at the jth node in level $\mathcal{S}_{(k+1)}$ can be modelled as a vector of symbols received over individual orthogonal channels corresponding to the retransmission of bit b from nodes in level $\mathcal{S}_k$. Thus the received signal can be expressed as:

$$\boldsymbol{r}_j^{(k+1)} = b\left(\boldsymbol{\beta}_j \odot \boldsymbol{\epsilon}^{(k)}\right) - 2b\left(\boldsymbol{\beta}_j \odot \boldsymbol{\epsilon}^{(k)} \odot \boldsymbol{e}^{(k)}\right) + \boldsymbol{w}_j \qquad (3)$$

where the operator $\odot$ represents element-wise multiplication, $\boldsymbol{\beta}_j$ is a $N \times 1$ vector (with i^{th} element β_{ji}) specifying

[1]Depending on the relay powers, this is actually not always possible; for further details see [5].

the link gains of all the orthogonal channels in the network, $\{\boldsymbol{w}_j\}_i \overset{iid}{\sim} \mathcal{CN}(0, N_o)$, and $\boldsymbol{\epsilon}^{(k)}$ and $\boldsymbol{e}^{(k)}$ are $N \times 1$ vectors whose i^{th} element is specified by

$$\epsilon_i^{(k)} = \begin{cases} 1 & \text{if the } i^{th} \text{ node} \in \mathcal{S}_k, \\ 0 & \text{otherwise} \end{cases}$$

$$e_i^{(k)} = \begin{cases} 1 & \text{if the } i^{th} \text{ node in } \mathcal{S}_k \text{ makes an error} \\ 0 & \text{otherwise.} \end{cases}$$

For a given network deployment $\boldsymbol{\epsilon}^{(k)}$ is known a priori if the node clustering is known. However, $\boldsymbol{\epsilon}^{(k)}$ is random since the deployment is random. As mentioned in Section 2, we assume that node j knows the combined vector $\boldsymbol{\beta}_j \odot \boldsymbol{\epsilon}^{(k)}$ without error.

Vector $\boldsymbol{e}^{(k)}$, which indicates the nodes that make an error in $\mathcal{S}_k$, is random and unknown to the next level. Information concerning the statistics of the error vector $\boldsymbol{e}^{(k)}$ should be incorporated in the construction of the Maximum-Likelihood detectors at the receivers. However, for ease of analysis we consider a suboptimal receiver that combines the observations in $\boldsymbol{r}_j^{(k+1)}$ using a Maximal-Ratio Combining (MRC) receiver. Hence, our analysis of the error propagation is pessimistic. The MRC detector rule is [10]:

$$\Re\left\{\left(\boldsymbol{\beta}_j \odot \boldsymbol{\epsilon}^{(k)}\right)^H \cdot \boldsymbol{r}_j^{(k+1)}\right\} \gtrless 0. \quad (4)$$

Let us define

$$z_j^{(k+1)} \triangleq \left\|\boldsymbol{\beta}_j \odot \boldsymbol{\epsilon}^{(k)}\right\|^2 = \sum_{i=1}^{N} |\beta_{ji}|^2 \epsilon_i^{(k)}, \quad (5)$$

$$v_j^{(k+1)} \triangleq \left\|\boldsymbol{\beta}_j \odot \boldsymbol{\epsilon}^{(k)} \odot \boldsymbol{e}^{(k)}\right\|^2 = \sum_{i=1}^{N} |\beta_{ji}|^2 \epsilon_i^{(k)} e_i^{(k)}. \quad (6)$$

Then the bit error rate of the j^{th} node in level L_{k+1} is

$$P\left(e_j^{(k+1)} = 1 | \mathcal{A}\right) = Q\left(\frac{z_j^{(k+1)} - 2v_j^{(k+1)}}{\sqrt{z_j^{(k+1)} \frac{N_o}{2}}}\right) \quad (7)$$

where $\mathcal{A} = \boldsymbol{\beta}_j$, $\boldsymbol{\epsilon}^{(k)}$, and $\boldsymbol{e}^{(k)}$ for the previous k.

These are the basic equations needed to analyze how the error propagates through the network; which essentially corresponds to evaluating the statistics of the vector $\boldsymbol{\epsilon}^{(k)} \odot \boldsymbol{e}^{(k)}$. For different values of k, the vector $(\boldsymbol{\epsilon}^{(k)} \odot \boldsymbol{e}^{(k)})$ is a Markov-chain whose statistics are cumbersome to analyze. Hence, to understand the behavior of the network we use a combination of asymptotic results that are valid when taking the limit for $N \to \infty$ while decreasing the transmit power $P_r \to 0$.

3.2 Continuum Network

In the continuum model we are interested in the behavior of *high density* networks with *constant sum-power*, i.e. the number of nodes, N, goes to infinity while the total relay power, $P_r N$, is fixed. This means that the number of nodes cooperating in each level of the cooperative transmission increases to infinity, however, the power density in each level remains finite.

We make use of the following assumptions in our analysis: (a) The positions of all nodes (x_i, y_i) are independent and identically distributed (i.i.d.), (b) The small scale fading coefficients α_{ji} have unit variance the phases θ_{ji} are i.i.d. uniform $[0, 2\pi]$, (c) d_{ji}, α_{ji}, and θ_{ji} are statistically independent $\forall j, i$, and (d) $\boldsymbol{w}_j$ is AWGN with zero mean and variance N_o.

In force of all the assumptions listed above, $\boldsymbol{\epsilon}^{(k)}$ and $\boldsymbol{e}^{(k)}$ are statistically independent. Under the above stated conditions the following lemma holds:

Lemma 1. *Assume that the network has unit area. In the asymptote, as $N \to \infty$ and $P_r \to 0$, by fixing $\lim_{N\to\infty} \lim_{P_r \to 0} P_r N = \bar{P}_r$ for every level $k > 1$ we get the following relations:*

The probability that the j^{th} node, located at coordinates (x, y) belongs to level L_k is such that

$$\lim_{N\to\infty} E\{\epsilon_j^{(k)}\} = \pi^{(k)}(x, y) = \begin{cases} 1 & (x, y) \in L_k, \\ 0 & \textit{otherwise} \end{cases} \quad (8)$$

With probability 1, $z_j^{(k+1)}$ converges to a deterministic value:

$$\begin{aligned} \lim_{\substack{N\to\infty \\ P_r\to 0}} z_j^{(k+1)} &= \xi^{(k+1)}(x, y) \\ &= \bar{P}_r \iint l(x-u, y-v)\pi^{(k)}(x, y)dudv \\ &= \bar{P}_r \iint_{L_k} l(x-u, y-v)dudv \end{aligned} \quad (9)$$

where the integration in the middle step is over the entire network area and $l(x-u, y-v)$ is the path-loss attenuation function between the nodes at coordinates (x, y) in level $\mathcal{S}_{k+1}$ and (u, v) in level $\mathcal{S}_k$. By letting $\lim_{\substack{N\to\infty \\ P_r\to 0}} E\{e_i^{(k)}\} = \psi^{(k)}(x, y)$, the coefficient $v_j^{(k+1)}$ tends to the following integral with probability 1:

$$\begin{aligned} \lim_{\substack{N\to\infty \\ P_r\to 0}} v_j^{(k+1)} &= \nu^{(k+1)}(x, y) \quad (10) \\ &= \bar{P}_r \iint_{L_k} l(x-u, y-v)\psi^{(k)}(u, v)dudv, \end{aligned}$$

and for all $k \geq 2$:

$$\psi^{(k)}(x, y) = Q\left(\frac{\xi^{(k)}(x, y) - 2\nu^{(k)}(x, y)}{\sqrt{\frac{N_o}{2}\xi^{(k)}(x, y)}}\right). \quad (11)$$

The set of equations above can be solved starting for $k = 1$ by using:

$$\psi^{(1)}(x, y) = E_\gamma\left\{Q\left(\sqrt{\frac{2P_s}{N_0}\gamma l(x, y)}\right)\right\} \quad (12)$$

where γ is the square of the fading envelope $|\alpha_{ji}|e^{j\theta_{ji}}$.

Proof. For a given N and P_r, denote the mean of $z_j^{(k+1)}$ by $M_N^{(k+1)}$. The following holds for $k \geq 1$:

$$\begin{aligned} M_N^{(k+1)} &\triangleq \mathbb{E}_\alpha\left\{z_j^{(k+1)}\right\} = \sum_{i=1}^{N} \mathbb{E}_\alpha\left\{|\beta_{ji}|^2 \epsilon_i^{(k)}\right\} \\ &= \frac{\bar{P}_r}{N} \sum_{i=1}^{N} l(d_{ji})\mathbb{E}\left\{\epsilon_i^{(k)}\right\} \end{aligned} \quad (13)$$

Because the nodes are deployed randomly under a uniform distribution (see Theorem 1 in [6]) we get

$$\lim_{N\to\infty} M_N^{(k+1)} = \bar{P}_r \iint_{L_k} l(x-u, y-v)dudv \quad (14)$$

By the Law of Large Numbers (LLN), (9) follows. Now let:

$$\psi^{(k)}(x_i, y_i) = \mathbb{E}\left\{e^{(k)}(x_i, y_i)\right\} \tag{15}$$

Arguing similarly for $v_j^{(k+1)}$, $\bar{M}_N^{(k+1)} \triangleq \mathbb{E}_\alpha\{v_j^{(k+1)}\}$ is

$$\bar{M}_N^{(k+1)} = \frac{\bar{P}_r}{N}\sum_{i=1}^{N} l(x_j - u_i, y_j - v_i)\mathbb{E}\{\epsilon_i^{(k)}\}\mathbb{E}\{e_i^{(k)}\}$$

$$\lim_{N\to\infty} \bar{M}_N^{(k+1)} = \bar{P}_r \iint_{L_k} l(x-u, y-v)\psi^{(k)}(u,v)dudv.$$

By the LLN this leads to (10). Because $Q(x)$ is a continuous function, the two results above imply that for $k \geq 2$ equation (11) is valid. □

Hence, what Lemma 1 states is that the expected probability of error for each value of the coordinates (x, y) is governed by a set of non-linear difference-integral equations i.e. (9), (10), and (11). The functions $\psi^{(k)}(x, y)$ that solve the equations in each region L_k represent the network error *dynamics*. By definition, $0 \leq \psi^{(k)}(x, y) \leq 0.5$. To evaluate the exact behavior of the error propagation across the levels in the asymptotic regime one must numerically evaluate the solutions for these equations for every value of $k = 1, ..., K$ (see Sec. 5).

Next, we obtain analytical insight on the error performance by finding bounds for $\psi^{(k)}(x, y)$ for every $k = 1, \ldots, K$. Catastrophic error propagation can be avoided if it is possible to choose L_k and/or $\bar{P}_r$ so that $\max \psi^{(k)}(x, y) \leq \lambda < 0.5$. The following lemma shows when and how this is possible:

Corollary 1. *Assuming that all L_k cover an equal area, let:*

$$\alpha = \min_{(x,y)} \sqrt{\frac{2\bar{P}_r}{N_o}\iint_{L_k} l(x-u, y-v)dudv} \quad \forall k = 1, \ldots, K. \tag{16}$$

Under the same assumptions that lead to Lemma 1, the worst error probability reaches one of the fixed points i.e. $\max\left(\psi^{(\infty)}(x, y)\right) \to \lambda$*, that is a solution of:*

$$\lambda = Q(\alpha(1 - 2\lambda)) \tag{17}$$

A trivial fixed point that can be achieved $\forall\ \alpha$ is $\lambda = 0.5$. It is the only fixed point for $0 \leq \alpha \leq \sqrt{\frac{\pi}{2}}$ and it means that for α in the above range, catastrophic error propagation will occur with probability one. For $\alpha > \sqrt{\frac{\pi}{2}}$ another fixed point exists[2]*:*

$$Q(\alpha) < \lambda < \frac{1}{2} - \frac{1}{2}\sqrt{-\frac{2}{\alpha^2}\ln\sqrt{\frac{\pi}{2\alpha^2}}}.$$

If $\psi^{(1)}(x, y)$ is smaller than $\frac{1}{2} - \frac{1}{2}\sqrt{-\frac{2}{\alpha^2}\ln\sqrt{\frac{\pi}{2\alpha^2}}}$, the worst error performance will be bounded by the fix point of (17).

Proof. Substituting the expressions for $\xi^{(k)}(x, y)$ and $\nu^{(k)}(x, y)$ into (11) it is not difficult to see that

$$\max\left(\psi^{(k)}(x, y)\right) \leq Q\left(\alpha\left[1 - 2\max(\psi^{(k-1)}(x, y))\right]\right) \tag{18}$$

[2]Note that for $\alpha \gg 1$, $\lambda \approx Q(\alpha)$ is a fixed point.

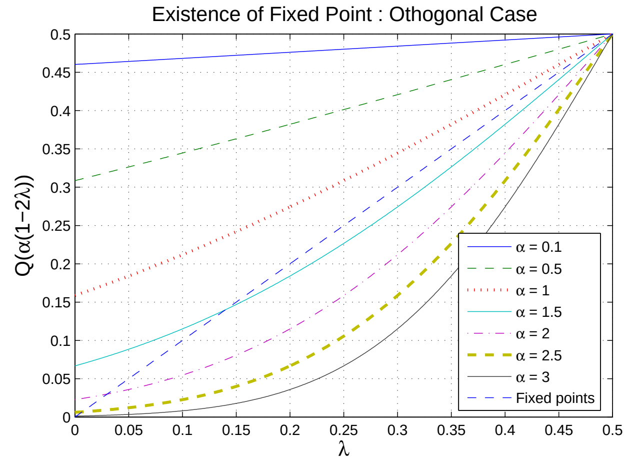

Figure 1: λ has a fixed point for some values of $Q(\alpha)$

If we replace $\max \psi^{(k-1)}(x, y)$ and $\max \psi^{(k)}(x, y)$ with λ_{k-1} and λ_k respectively in (18), we get:

$$\lambda_k \leq Q(\alpha(1 - 2\lambda_{k-1})) \tag{19}$$

which implies that λ will tend to be limited to at most one of the exact solutions of the equation (19).

When $\lambda = 0.5$ the equality is met no matter what α is. Because $0 \leq BER \leq 0.5$ we only need to analyze if another solution exists in this range. The function $Q(\alpha(1 - 2\lambda))$ is monotonically increasing in this range and is concave because:

$$\frac{dQ(\alpha(1-2\lambda))}{d\lambda} = \sqrt{\frac{2\alpha^2}{\pi}}e^{-\frac{\alpha^2(1-2\lambda)^2}{2}} \geq 0.$$

For $\lambda = 0$ the function $Q(\alpha) > 0$ for any finite α. Hence, $Q(\alpha(1-2\lambda))$ and the straight line have another intersection point for $\lambda < 0.5$ if α is such that the point where $Q(\alpha(1 - 2\lambda))$ has tangent 1 is less than $\lambda = 0.5$ (see Figure 1). The value of $\alpha_\star$ for which the phase transition happens is:

$$\left.\frac{dQ(\alpha_\star(1-2\lambda))}{d\lambda}\right|_{\lambda=0.5} = \sqrt{\frac{2\alpha_\star^2}{\pi}} = 1 \quad \Rightarrow \quad \alpha_\star = \sqrt{\frac{\pi}{2}}.$$

For all $\alpha > \alpha_\star = \sqrt{\frac{\pi}{2}}$, two intersection points exist: one is still at $\lambda = 0.5$ and the other one is for a $\lambda \geq Q(\alpha)$.

The second intersection point also has to be below the point $\lambda_\star$ where $\frac{dQ(\alpha(1-2\lambda_\star))}{d\lambda} = 1$ where for $\alpha > \alpha_\star$ it holds that $Q(\alpha(1 - 2\lambda_\star)) < \lambda_\star$. Thus for $\alpha > \sqrt{\frac{\pi}{2}}$, $Q(\alpha) \leq \lambda \leq \lambda_\star$. The value of $\lambda_\star$ is calculated easily, by solving $\sqrt{2\alpha^2/\pi}\exp(-\frac{\alpha^2(1-2\lambda_\star)^2}{2}) = 1$ which gives $\lambda_\star = 0.5 - 0.5\sqrt{-\frac{2}{\alpha^2}\ln\sqrt{\frac{\pi}{2\alpha^2}}}$. □

3.3 Comparison with a Non-Cooperative Network

For $\alpha > \sqrt{\frac{\pi}{2}}$, Corollary 1 shows that the BER over each level of the cooperative network is going to be bounded. Suppose that the cooperative physical layer is replaced by a single node at the edge of each level that can transmit at the total power of an entire level. Denoting the best point at each level to forward the message by (x_k, y_k), the average

BER from level to level is

$$P_{k,k-1} = \mathbb{E}_{\gamma_k}\left\{Q\left(\frac{2\bar{P}_r\,|S_k|}{N_o}\gamma_k l(x_k - x_{k-1}, y_k - y_{k-1})\right)\right\}$$

where γ_k is the square of the fading envelope of the 'hop.' Therefore, the BER for the multi-hop network is

$$BER = 1 - \prod_{k=1}^{\mathcal{K}} (1 - P_{k,k-1}) \approx \mathcal{O}(\mathcal{K}) \quad (20)$$

because $BER \geq 1 - (1 - \min_k (P_{k,k-1}))^{\mathcal{K}} = \mathcal{O}(\mathcal{K})$ if $\min_k(P_{k,k-1}) \ll 1$. This indicates that eventually the error will grow to 0.5. The comparison made above is not entirely fair because the cooperative network we analyzed uses infinite bandwidth whereas the non-cooperative multi-hop network does not. Below we show that this asymptotic bandwidth expansion is unnecessary. The low achievable bounds on the error dynamics we have found above for the cooperative scheme using orthogonal channels are shared by a scheme that utilizes an amount of bandwidth comparable to that of the multi-hop scheme.

4. NON-ORTHOGONAL TRANSMISSIONS

4.1 Random Finite Network

The problem setup is the same as that described for orthogonal transmission in Section 3.1 except that the nodes in the same level are no longer assigned orthogonal channels for their transmissions. Hence, for every transmitted binary symbol there will be only one received coefficient per level. Using similar notation and the same definitions for $\boldsymbol{\beta}_j$, $\boldsymbol{\epsilon}^{(k)}$ and $\boldsymbol{e}^{(k)}$ as in the orthogonal transmission case, the sample received by the j^{th} node in level L_{k+1} which corresponds to a certain binary symbol b is:

$$r_j^{(k+1)} = b\left(\boldsymbol{\beta}_j^T \cdot \boldsymbol{\epsilon}^{(k)}\right) - 2b\boldsymbol{\beta}_j^T \cdot \left(\boldsymbol{\epsilon}^{(k)} \odot \boldsymbol{e}^{(k)}\right) + w_j^{(k+1)}. \quad (21)$$

Note that $r_j^{(k+1)}$ is a now a scalar quantity and $w_j^{(k+1)}$ is the sole noise sample that distorts the reception of binary symbol b over the shared cooperative channel. Let us define

$$z_j^{(k+1)} \triangleq \boldsymbol{\beta}_j^T \cdot \boldsymbol{\epsilon}^{(k)} \quad \text{and} \quad v_j^{(k+1)} \triangleq \boldsymbol{\beta}_j^T \cdot \left(\boldsymbol{\epsilon}^{(k)} \odot \boldsymbol{e}^{(k)}\right).$$

Again, assume that during the training phase the receivers estimate $z_j^{(k+1)}$ accurately. The coherent detector uses the following decision rule

$$\Re\left\{\left(z_j^{(k+1)}\right)^* \cdot r_j^{(k+1)}\right\} \gtrless 0. \quad (22)$$

The bit error rate of the j^{th} node in level L_{k+1} for $k \geq 2$ is

$$P\left(e_j^{(k+1)} = 1|\mathcal{A}\right) = Q\left(\frac{\left|z_j^{(k+1)}\right|^2 - 2\Re\left\{z_j^{*(k+1)} \cdot v_j^{(k+1)}\right\}}{\sqrt{\frac{N_o}{2}\left|z_j^{(k+1)}\right|^2}}\right) \quad (23)$$

where $\mathcal{A} = \boldsymbol{\beta}_j$, $\boldsymbol{\epsilon}^{(k)}$, and $\boldsymbol{e}^{(k)}$ for the previous k.

Once again, rather than analyzing the Markov-chain ($\boldsymbol{\epsilon}^{(k)} \odot \boldsymbol{e}^{(k)}$) for a finite network we analyze its asymptote in the following section.

4.2 Continuum Network

The following lemma holds for a network of unit area.

LEMMA 2. *In the asymptote, as $N \to \infty$ and $P_r \to 0$, by fixing $\lim_{N\to\infty} \lim_{P_r\to 0} P_r N = \bar{P}_r$ we get the following relations - Let $\lim_{\substack{N\to\infty \\ P_r\to 0}} \mathbb{E}\left\{e_j^{(k)}\right\} = \psi^{(k)}(x,y)$. The probability that the j^{th} node belongs to level L_k is:*

$$\lim_{N\to\infty} E\{\epsilon_j^{(k)}\} = \pi^{(k)}(x,y) = \begin{cases} 1 & (x,y) \in L_k, \\ 0 & otherwise \end{cases} \quad (24)$$

The coefficients $\lim_{N\to\infty} \lim_{P_r\to 0}(z_j^{(k+1)}, v_j^{(k+1)})$ converge in distribution to a zero mean jointly Gaussian circularly symmetric random variable such that:

$$\begin{aligned} VAR\left(\lim_{\substack{N\to\infty \\ P_r\to 0}} z_j^{(k+1)}\right) &= \xi^{(k+1)}(x,y) \quad (25) \\ &= \bar{P}_r \iint_{L_k} l(x-u, y-v)dudv \end{aligned}$$

$$\begin{aligned} VAR\left(\lim_{\substack{N\to\infty \\ P_r\to 0}} v_j^{(k+1)}\right) &= \nu^{(k+1)}(x,y) \quad (26) \\ &= \bar{P}_r \iint_{L_k} l(x-u, y-v)\psi^{(k)}(u,v)dudv \end{aligned}$$

and their correlation coefficient is asymptotically:

$$\rho\left(\lim_{\substack{N\to\infty \\ P_r\to 0}} (z_j^{(k+1)}, v_j^{(k+1)})\right) = \sqrt{\frac{\nu^{(k+1)}(x,y)}{\xi^{(k+1)}(x,y)}} \quad (27)$$

PROOF. See Appendix A. □

The following corollary is used to perform the numerical analysis for the case of non-orthogonal channels.

COROLLARY 2. *The average error rate (23) of level $k \geq 2$ is bounded as follows:*

$$\psi^{(k)}(x,y) \leq \frac{1}{2} \frac{1}{\sqrt{1 + \frac{4}{N_o}\nu\left(1 - \frac{\nu}{\xi}\right)}} \cdot \frac{1}{1 + G\xi} \quad (28)$$

where $\nu = \nu^{(k)}(x,y)$ as in equation (26), $\xi = \xi^{(k)}(x,y)$ as in equation (25), and G is

$$G_{(x,y)}^{(k)} = \frac{\frac{1}{N_o}\left(1 - 2\frac{\nu_{(x,y)}^{(k)}}{\xi_{(x,y)}^{(k)}}\right)^2}{1 + \frac{4}{N_o}\nu_{(x,y)}^{(k)}\left(1 - \frac{\nu_{(x,y)}^{(k)}}{\xi_{(x,y)}^{(k)}}\right)} \quad (29)$$

The average error rate for $k = 1$ is given by (12).

PROOF. See Appendix B. □

We now present the main result of this section.

COROLLARY 3. *Let α be defined as in Corollary 1. Then $\lambda \triangleq \max \psi^{(k)}(x,y)$ tends to be the fixed point of the following equation:*

$$\lambda_k \leq \frac{0.5}{\sqrt{1 + 0.5\alpha^2}} \cdot \sqrt{1 - (1 - 2\lambda_{k-1})^2 + \frac{1}{1 + \frac{\alpha^2}{2}}} \quad (30)$$

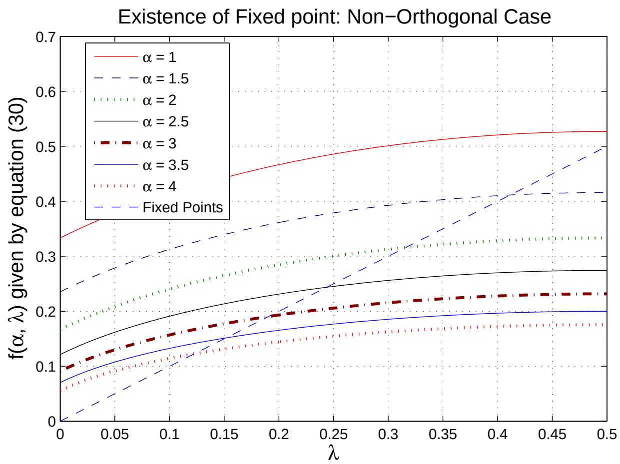

Figure 2: λ has a fixed point for some values of α

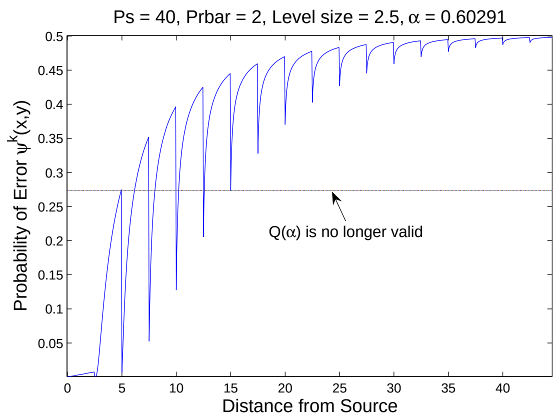

Figure 4: Error propagation in Orthogonal Transmission. When $\alpha < \sqrt{\pi/2}$ there is catastrophic error propagation.

For $\alpha^2/2 > 0.618$, this equation has only one positive fixed point in the range $0 \leq \lambda \leq 0.5$. This fixed point is:

$$\lambda = \frac{1 + \sqrt{2 + \left(1 + \frac{\alpha^2}{2}\right)^{-1}}}{2\left(2 + \frac{\alpha^2}{2}\right)} \tag{31}$$

For $\alpha^2/2 \leq 0.618$, the upper bound on λ is 0.5 rather than equation (31) (see Figure 2). For $\alpha \gg 1$, $\lambda \sim \frac{1+\sqrt{2}}{\alpha^2}$

PROOF. See Appendix C. □

5. NUMERICAL RESULTS

For both the orthogonal and non-orthogonal transmission cases we numerically solve the recursive equations that characterize the error propagation in each level. Setting the position of the source to be the origin, the plots below show the average BER $\psi^{(k)}(x,0)$ of nodes located at coordinates $(x,0)$. For both types of transmission, the BER for level 1 is given by equation (12). For orthogonal transmissions the BER for subsequent levels is determined by solving equation (11).

The numerical evaluation shows that the analytical results presented in Corollary 1 provide accurate values for the fixed point. Figure 3 shows that for values of $\bar{P}_r$ and level size such that $\alpha > \sqrt{\pi/2}$, the error propagation can be controlled and the worst error probability, $\psi^k(x,y)$, is close to $Q(\alpha)$. On the other hand, when $\alpha < \sqrt{\pi/2}$ error propagation is catastrophic. This is shown in Figure 4.

In Figure 5 we plot the average BER as given by equation (28). Again, if $\bar{P}_r$ and level size are set to values such that $\alpha^2/2 > 0.618$ then the BER can be controlled. It is observed that upper bound predicted by Corollary 3 is larger than the fixed point obtained. This follows from the fact that Corollary 3 is obtained by upper bounding (28).

In Figures 6 and 7 we plot the BER for the orthogonal and non-orthogonal cases as given by equations (7) and (23) for different network node densities. In each case the BER is averaged over several trials to average out the effects of random Rayleigh fading and the random locations of the nodes. The figures indicate that the results are consistent with the BER bounds predicted by equations (11) and (28).

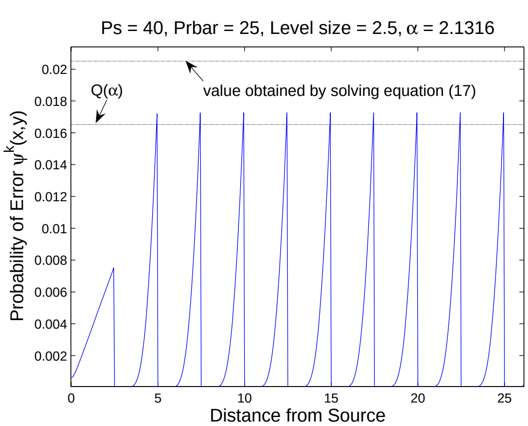

Figure 3: Error propagation in Orthogonal Transmission. Each tooth of the saw-tooth-like function represents the BER in a separate level.

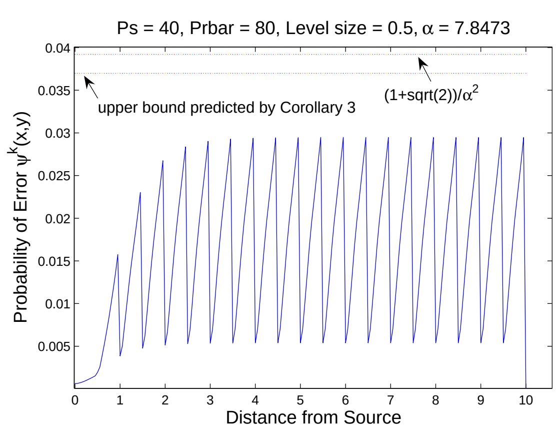

Figure 5: Error propagation in Non-Orthogonal Transmission. Each tooth of the saw-tooth-like function represents the BER in a separate level.

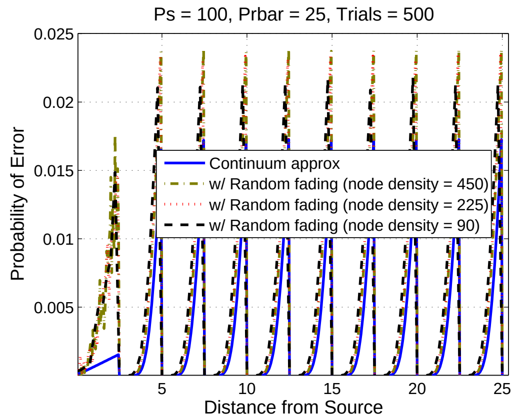

Figure 6: Error propagation in Orthogonal Transmission for different network densities.

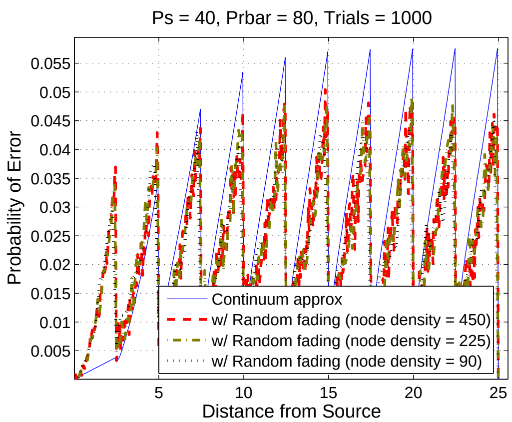

Figure 7: Error propagation in Non-Orthogonal Transmission for different network densities.

6. FINAL REMARKS

The asymptotic results in Corollaries 1 and 3 show that not only is the error bounded by the fixed point but that the fixed point, λ, is a nice function of the parameter α^2; where α can be interpreted as the cumulative SNR for each level. It is evident from Figure 1 that for the orthogonal transmission scheme the lower bound on the BER, $Q(\alpha)$, becomes tight for $\alpha \gg 1$. This scheme requires infinite bandwidth for gaining infinite diversity over the wireless medium and it nearly matches the performance of a cooperative transmission scheme over a deterministic channel i.e. an AWGN channel without fading. In contrast, the non-orthogonal transmission scheme cannot avoid fading because all cooperating nodes transmit asynchronously with respect to their carrier-phases. Here, the BER $\rightarrow 1/\alpha^2$ which is directly proportional to $1/SNR$ as one would intuitively guess. This result should not discourage us from using the non-orthogonal transmission scheme because in terms of providing a controllable average BER performance the non-orthogonal scheme is as effective as the orthogonal scheme. Furthermore, it does not require a bandwidth expansion because it allows transmissions from all cooperating nodes to effectively *collide*, to use networking jargon. The non-orthogonal scheme is also much simpler to use in a distributed network setting. To use the orthogonal scheme, the orthogonal channels have to somehow be assigned between the cooperating nodes. For the non-orthogonal scheme, on the other hand, we only require the synchronization of the nodes to the beginning of the packet they received. By using multi-carrier and/or spread spectrum transmission in each level, moderate asynchronism among the cooperating nodes can be tolerated. In this case, we expect to observe a similar scaling law for the error rates even if the transmissions of the nodes in a level are not perfectly synchronized as required by our simple model.

7. APPENDIX

7.1 Proof of Lemma 2

The proof derives the joint characteristic function of the random variables $\lim_{\substack{N\to\infty \\ P_r\to 0}} (z_j^{(k+1)}, v_j^{(k+1)})$. In the proof the function $\Phi_X(f)$ will be always the characteristic function of the random variable X, i.e. $\Phi_X(f) = \mathbb{E}\{e^{-j2\pi f X}\}$:

$$\Phi(f_1,f_2) = \mathbb{E}\left\{\lim_{\substack{N\to\infty \\ P_r\to 0}} e^{-j2\pi(f_1 z_j^{(k+1)} + f_2 v_j^{(k+1)})}\right\} \tag{32}$$

$$= \mathbb{E}\left\{\lim_{\substack{N\to\infty \\ P_r\to 0}} \prod_{i=1}^{N} \Phi_{e^{j\theta_{ji}}}\left(\sqrt{\frac{\bar{P}_r \alpha_{ji} l(d_{ji})}{N}}\epsilon_i^{(k)}(f_1 + f_2 e_i^{(k)})\right)\right\}$$

$$= \mathbb{E}\left\{\lim_{\substack{N\to\infty \\ P_r\to 0}} \prod_{i=1}^{N}\left(1 - \frac{4\pi^2 \bar{P}_r l(d_{ji})\alpha_{ji}\epsilon_i^{2(k)}(f_1+f_2 e_i^{(k)})^2}{N} + o\left(\frac{1}{N}\right)\right)\right\}$$

$$= \mathbb{E}\left\{\lim_{N\to\infty} exp\{-\frac{1}{N}\sum_{i=1}^{N} 4\pi^2 \bar{P}_r l(d_{ji})\alpha_{ji}\epsilon_i^{2(k)\}(f_1+f_2 e_i^{(k)})^2}\right\}$$

Therefore if we take the expectation of the term inside the exponent, as argued before we can show that:

$$M_N = \mathbb{E}\left\{\frac{1}{N}\sum_{i=1}^{N} 4\pi^2 \bar{P}_r l(d_{ji})\alpha_{ji}\epsilon_i^{2(k)}(f_1+f_2 e_i^{(k)})^2\right\}$$

$$\lim_{N\to\infty} M_N = 4\pi^2 \bar{P}_r \iint_{L_k} l(x-u, y-v)\left(f_1^2 + \right.$$

$$\left. + 2 f_1 f_2 \psi^{(k)}(u,v) + f_2^2 \psi^{(k)}(u,v)\right) dudv.$$

Let us call X_N the random variable:

$$X_N = \frac{1}{N}\sum_{i=1}^{N} 4\pi^2 \bar{P}_r l(d_{ji})\alpha_{ji}\epsilon_i^{2(k)}(f_1+f_2 e_i^{(k)})^2,$$

which for the Law of Large Numbers is such that, for an arbitrarily small $\delta > 0$: $\lim_{N\to\infty} P(|M_N - X_N| > \delta) = 0$. Clearly: $\Phi(f_1,f_2) = e^{-\lim_{N\to\infty} M_N} - o(\delta)$. The expression of $\Phi(f_1,f_2)$ is that of the characteristic function of two jointly Gaussian random variables and by inspection it can be verified that the parameters of the quadratic form $\lim_{N\to\infty} M_N$ give the variances and correlation coefficients in Lemma 2.

7.2 Proof of Corollary 2

In the following proof we drop the sub/super-scripts in expressions where there is no ambiguity for notational convenience. From equation (23),

$$\mathbb{E}\left\{P\left(e_j^{(k)}=1\right)\right\}=\psi^{(k)}(x,y) \tag{33}$$

$$\leq \frac{1}{2}\mathbb{E}\left\{\exp\left[-\frac{\left|z_{(x,y)}^{(k)}\right|^2}{N_o}\left(1-\frac{2\Re\left\{z_{(x,y)}^{*(k)}v_{(x,y)}^{(k)}\right\}}{\left|z_{(x,y)}^{(k)}\right|^2}\right)^2\right]\right\}$$

$$= \frac{1}{2}\mathbb{E}_z\mathbb{E}_{v|z}\exp\left[-\frac{\left|z_{(x,y)}^{(k)}\right|^2}{N_o}\left(1-\frac{2\Re\left\{z_{(x,y)}^{*(k)}v_{(x,y)}^{(k)}\right\}}{\left|z_{(x,y)}^{(k)}\right|^2}\right)^2\right]$$

Let $s=-\left|z_{(x,y)}^{(k)}\right|^2/N_o$ and $t=1-2\Re\left\{z_{(x,y)}^{*(k)}v_{(x,y)}^{(k)}\right\}/\left|z_{(x,y)}^{(k)}\right|^2$. By utilizing the Gaussian probability density function of $\lim_{\substack{N\to\infty \\ P_r\to 0}}(z_j^{(k+1)},v_j^{(k+1)})$ found in lemma 2, it follows that the conditional distribution of t, given that $z_{(x,y)}^{(k)}$ is known, is $t\sim\mathcal{CN}(\mu,\sigma^2)$ where $\mu=\left(1-2\nu_{(x,y)}^{(k)}/\xi_{(x,y)}^{(k)}\right)$ and $\sigma^2=\left(1-\nu_{(x,y)}^{(k)}/\xi_{(x,y)}^{(k)}\right)2\nu_{(x,y)}^{(k)}/\left|z_{(x,y)}^{(k)}\right|^2$. It is easily shown that

$$\mathbb{E}\left\{e^{st^2}\right\}=\frac{1}{\sqrt{1-2s\sigma^2}}e^{\frac{s\mu^2}{1-2s\sigma^2}}.$$

By substituting these expressions into equation (33) we thus get,

$$\psi^{(k)}(x,y)\leq\frac{1}{2}\mathbb{E}_{z_{(x,y)}^{(k)}}\left\{e^{st^2}\right\}$$

$$=\frac{1}{2}\mathbb{E}_z\left\{\frac{1}{\sqrt{1+\frac{4}{N_o}\nu\left(1-\frac{\nu}{\xi}\right)}}\exp\left[\frac{-\frac{|z|^2}{N_o}\left(1-2\frac{\nu}{\xi}\right)^2}{1+\frac{4}{N_o}\nu\left(1-\frac{\nu}{\xi}\right)}\right]\right\}$$

$$=\frac{1}{2}\frac{1}{\sqrt{1+\frac{4}{N_o}\nu\left(1-\frac{\nu}{\xi}\right)}}\mathbb{E}_z\left\{\exp\left(-|z|^2G\right)\right\}$$

$$=\frac{1}{2}\frac{1}{\sqrt{1+\frac{4}{N_o}\nu\left(1-\frac{\nu}{\xi}\right)}}\cdot\frac{1}{1+G\xi} \tag{34}$$

where we have used that $\left|z_{(x,y)}^{(k)}\right|^2\sim exp\left(\frac{1}{\xi_{(x,y)}^{(k)}}\right)$ and

$$G_{(x,y)}^{(k)}=\frac{\frac{1}{N_o}\left(1-2\frac{\nu_{(x,y)}^{(k)}}{\xi_{(x,y)}^{(k)}}\right)^2}{1+\frac{4}{N_o}\nu_{(x,y)}^{(k)}\left(1-\frac{\nu_{(x,y)}^{(k)}}{\xi_{(x,y)}^{(k)}}\right)} \tag{35}$$

7.3 Proof of Corollary 3

After plugging-in the expression for $G^k(x,y)$, equation (28) [c.f. Corollary 2] can be rearranged as follows:

$$\psi^k(x,y)\leq\frac{0.5}{\sqrt{1+\frac{\xi}{N0}}}\cdot\sqrt{1-\left(1-\frac{\nu}{\xi}\right)^2+\frac{1}{1+\frac{\xi}{N0}}\left(1-\frac{\nu}{\xi}\right)^2} \tag{36}$$

We know that for any given level the following inequalities are true:

$$0\leq\frac{\nu^k(x,y)}{\xi^k(x,y)}\leq\lambda_k\leq 0.5\Rightarrow\left(1-\frac{2\nu^k(x,y)}{\xi^k(x,y)}\right)\leq 1, \tag{a}$$

$$1-\left(1-\frac{2\nu^k(x,y)}{\xi^k(x,y)}\right)^2\leq 1-(1-2\lambda_k)^2, \tag{b}$$

$$\frac{1}{1+\frac{\xi^k(x,y)}{N0}}\leq\frac{1}{1+\frac{\alpha^2}{2}}, \tag{c}$$

and from (a) and (c),

$$\frac{1}{1+\frac{\xi^k(x,y)}{N0}}\left(1-\frac{2\nu^k(x,y)}{\xi^k(x,y)}\right)^2\leq\frac{1}{1+\frac{\alpha^2}{2}} \tag{d}$$

By applying (a)-(d) to equation (36) we get:

$$\psi^k(x,y)<\lambda_k$$

$$\leq\frac{0.5}{\sqrt{1+\frac{\alpha^2}{2}}}\sqrt{1-(1-2\lambda_{k-1})^2+\frac{1}{1+\frac{\alpha^2}{2}}} \tag{37}$$

By squaring both sides of the above we obtain a second order equation:

$$\lambda^2\left(2+\frac{\alpha^2}{2}\right)-\lambda-\frac{1}{4\left(1+\frac{\alpha^2}{2}\right)}=0 \tag{38}$$

This equation has a real positive and a real negative root. Since λ cannot be negative, the only solution is as indicated in Corollary 3. The solution for the fixed point exceeds 0.5 if $\alpha^2/2<0.618$. Hence, in this regime the bound becomes too loose to be meaningful.

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